REDUCTION OF PRE-HAMILTONIAN ACTIONS

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Abstract. We prove a reduction theorem for the tangent bundle of a Poisson manifold (M, π) endowed with a pre-Hamiltonian action of a Poisson Lie group (G, π_G). In the special case of a Hamiltonian action of a Lie group, we are able to compare our reduction to the classical Marsden-Ratiu reduction of M. If the manifold M is symplectic and simply connected, the reduced tangent bundle is integrable and its integral symplectic groupoid is the Marsden-Weinstein reduction of the pair groupoid $M \times \bar{M}$.

1. Introduction

Reduction procedures for manifolds with symmetries are known in many different settings. A quite general approach, whose origin traces back to the ideas of Cartan [8], was considered in [32] and then generalized in [2, 3]. In this approach, the reduction of a symplectic manifold (M, ω) is intended as a submersion $\rho : N \to M_{red}$ of an immersed submanifold $i : N \hookrightarrow M$ onto another symplectic manifold (M_{red}, ω_{red}) such that $i^*\omega = \rho^*\omega_{red}$. In particular M_{red} might be the space of leaves of the characteristic distribution of $i^*\omega$. However, the most famous result is the one provided by Marsden and Weinstein [28] in the special case where the submanifold N consists of a level set of a momentum map associated to the Hamiltonian action a of Lie group. One of the possible generalizations has been introduced by Lu [23] and concerns actions of Poisson Lie groups on symplectic manifolds. Afterwards, the case of Poisson Lie groups acting on Poisson manifolds has been studied in [13]. It is also worth to mention the case of Manin pairs (that include Dirac and Poisson manifolds as special cases) treated in [6] and the super-geometric setting studied in [10]. In this paper we consider the case of Poisson Lie groups acting on Poisson manifolds. Such actions appear naturally in the study of R matrices and they encode the hidden symmetries of classical integrable systems. An action of a Poisson Lie group (G, π_G) is said to be Poisson Hamiltonian if it is generated by an equivariant momentum map $J : M \to G^*$. We shall focus on a further generalization of Poisson Hamiltonian actions. The main idea, introduced by Ginzburg in [19], is to consider only the infinitesimal version of the equivariant momentum map studied by Lu. An action induced by such an infinitesimal momentum map is what we call pre-Hamiltonian. Any Poisson Hamiltonian action is pre-Hamiltonian. Conversely, we show that any pre-Hamiltonian action is a Poisson action. However, pre-Hamiltonian actions are strictly more general of Hamiltonian actions, as shown in [19] where concrete examples of pre-Hamiltonian actions which are not Poisson Hamiltonian were provided.

We obtain a reduction theorem for the tangent bundle of a Poisson manifold endowed with a pre-Hamiltonian action of a Poisson Lie group. First, we show that a pre-Hamiltonian action of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π) defines a coisotropic submanifold C of the tangent bundle TM. Hence, we

can build up a reduced space by using the theory of coisotropic reduction. In fact, given a coisotropic submanifold C of TM, the associated characteristic distribution allows us to define a leaf space $C/\sim$, which we denote by $(TM)_{red}$. The coisotropic submanifold C is defined by means of a map $\tilde{\varphi}$ from $\mathfrak{g}$ to the space of 1-forms on M which preserves the Lie algebra structures and is a cochain map. We obtain the following

Theorem 1.1. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson Lie group G on a Poisson manifold (M, π). Then the reduced tangent bundle $(TM)_{red}$ carries a Poisson structure. Moreover $(TM)_{red} \to M/G$ is a Lie algebroid.*

The proof uses the theory of coisotropic reduction, the Tulczyjew's isomorphisms [34, 35] and the theory of tangent derivations [30].

Given a pre-Hamiltonian action, we compute explicitly the infinitesimal generator of the tangent lift of the action, as can be seen in the following

Theorem 1.2. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson Lie group with infinitesimal momentum map $\tilde{\varphi}$.*

(i) *The infinitesimal generator φ^T of the tangent lift of Φ is given by*

$$\varphi^T(\xi) = X_{i_T \tilde{\varphi}_\xi} + \pi^\sharp_{TM} \circ i_T \mathrm{d}\, \tilde{\varphi}_\xi.$$

(ii) *If for each $\xi \in \mathfrak{g}$, one has $\mathrm{d}\, \tilde{\varphi}_\xi = 0$, then the lifted (infinitesimal) action on (TM, π_{TM}) is Hamiltonian, with fiberwise-linear momentum map defined by $c_\xi = i_T \tilde{\varphi}_\xi$.*

Note that in the special case of a symplectic action on a symplectic manifold we always obtain a Hamiltonian action on the tangent bundle and hence a Marsden-Weinstein reduction of TM can be performed.

In order to relate our reduction to the classical Marsden-Ratiu reduction, we consider the particular case of a Hamiltonian action and prove the following

Theorem 1.3. *Let (M, π) be a Poisson manifold endowed with a Hamiltonian action of a Lie group G and $0 \in \mathfrak{g}^*$ a regular value of the momentum map J. Then there is a connection dependent isomorphism of vector bundles*

$$(TM)_{red}|_{M_{red}} \cong T(M_{red}) + \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the associated bundle to the principal bundle $J^{-1}(0) \to J^{-1}(0)/G$ by the adjoint action of G on $\mathfrak{g}$.

Furthermore, by using [17, 16] we provide an interpretation of the reduced tangent bundle in terms of symplectic groupoids. In particular, we consider the case of a symplectic action of a Lie group G on a symplectic manifold M. On the one hand, we show that in this case the lifted action on the tangent bundle TM is Hamiltonian so that we obtain a reduced tangent bundle $(TM)_{red}$ which is a symplectic manifold. On the other hand, it follows from [29] that the symplectic action on (M, ω) can be lifted to a Hamiltonian action on the corresponding symplectic groupoid that can be identified with the fundamental groupoid $\Pi(M) \rightrightarrows M$ of M. This implies that the symplectic groupoid can be reduced via Marsden-Weinstein procedure to a new symplectic groupoid $(\Pi(M))_{red} \rightrightarrows M/G$. We prove that in this case our reduced tangent bundle $(TM)_{red}$ is the Lie algebroid corresponding to the reduced symplectic groupoid $(\Pi(M))_{red}$. More precisely,

Theorem 1.4. *Given a free and proper symplectic action of a Lie group G on a symplectic manifold (M,ω), we have*

$$A((\Pi(M))_{red}) \cong (TM)_{red}.$$

If M is simply connected, this is just the reduction of the pair groupoid $M \times \bar{M}$.

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2. Hamiltonian actions and coisotropic reduction

In this section we recall some well-known results regarding reduction procedures for Hamiltonian actions and for the more general case of coisotropic submanifolds which will be used in the following sections.

Let G be a Lie group and (M,π) a Poisson manifold. An action $\Phi : G \times M \to M$ is said to be **canonical** if it preserves the Poisson structure π on M. Let $\varphi : \mathfrak{g} \to \Gamma(TM)$ be the infinitesimal generator of the action. In order to perform a reduction we need to introduce the notion of momentum map.

Definition 2.1. A **momentum map** for a canonical action of G on M is a map $J : M \to \mathfrak{g}^*$ such that it generates the action by

$$\varphi(\xi) = \pi^\sharp(\mathrm{d}\, J_\xi),$$

where $J_\xi : M \to \mathbb{R}$ is defined by $J_\xi(p) = \langle J(p), \xi\rangle$, for any $p \in M$ and $\xi \in \mathfrak{g}$.

A momentum map $J : M \to \mathfrak{g}^*$ is said to be **equivariant** if it is a Poisson map, where $\mathfrak{g}^*$ is endowed with the so-called Lie Poisson structure [7, Sec. 3]. Finally, a canonical action is said to be **Hamiltonian** if it is generated by an equivariant momentum map.

A reduction theorem for symplectic manifolds with respect to Hamiltonian group action was proven in [28]. It extends in a straightforward way to the case of Poisson manifolds which we now recall.

Theorem 2.2 ([27])**.** *Let (M,π) be a Poisson manifold endowed with a free and proper Hamiltonian action of a Lie group G and assume that $0 \in \mathfrak{g}^*$ is a regular value for the momentum map $J : M \to \mathfrak{g}^*$. Then the reduced space*

$$M_{red} = J^{-1}(0)/G$$

is a Poisson manifold.

Now we briefly review a more general procedure, called coisotropic reduction. The main idea, due to Weinstein [36], is that given a Poisson manifold and a coisotropic submanifold, one can always build up a reduced space. Some nice reviews of this theory can be found in [4], [5] and [9]. The reduction of a Poisson manifold with respect to a Hamiltonian group action as well as coisotropic reduction can be recovered as special cases of reduction by distributions [27, 15, 21].

Let (M, π) be a Poisson manifold and $C \subseteq M$ a submanifold. We denote by

$$\mathscr{I}_C = \{f \in \mathscr{C}^\infty(M) : f|_C = 0\} \tag{2.1}$$

the multiplicative ideal of the Poisson algebra $\mathscr{C}^\infty(M)$. It is known that C is coisotropic if and only if $\mathscr{I}_C$ is a Poisson subalgebra. From now on, we assume that C is a regular closed submanifold, so we have the identification

$$\mathscr{C}^\infty(M)/\mathscr{I}_C \cong \mathscr{C}^\infty(C).$$

Assume that (M, π) is a Poisson manifold and $C \subseteq M$ is a coisotropic submanifold. From the properties of coisotropic manifolds, we know that there always exists a characteristic distribution on C, which is spanned by the Hamiltonian vector fields X_f associated to $f \in \mathscr{I}_C$. This distribution is integrable, so we can define the leaf space

$$M_{red} := C/\!\sim.$$

We assume that the corresponding foliation is simple, that is M_{red} is a smooth manifold and the projection map

$$p : C \to M_{red} \tag{2.2}$$

is a surjective submersion. The manifold M_{red} is called the **reduced manifold**. One can show that M_{red} is a Poisson manifold. More precisely, the following results hold (see [31, 33]).

Proposition 2.3. *Let (M, π) be a Poisson manifold and $C \subseteq M$ a coisotropic submanifold.*

(i) $\mathscr{B}_C := \{f \in \mathscr{C}^\infty(M) : \{f, \mathscr{I}_C\} \subseteq \mathscr{I}_C\}$ *is a Poisson subalgebra of* $\mathscr{A}$ *containing* $\mathscr{I}$*.*
(ii) $\mathscr{I}_C \subseteq \mathscr{B}_C$ *is a Poisson ideal and* $\mathscr{B}_C$ *is the largest Poisson subalgebra of* $\mathscr{C}^\infty(M)$ *with this feature*
(iii) $\mathscr{C}^\infty(M)_{red} := \mathscr{B}_C/\mathscr{I}_C$ *is a Poisson algebra.*

The relation between $\mathscr{C}^\infty(M_{red})$ and $\mathscr{C}^\infty(M)_{red}$ is given by the following theorem.

Theorem 2.4. *Let M be a Poisson manifold and C a closed regular coisotropic submanifold defining a simple foliation, so that*

$$p : C \to M_{red}$$

is a surjective submersion. Then there exists a Poisson structure on M_{red} such that $\mathscr{C}^\infty(M)_{red}$ and $\mathscr{C}^\infty(M_{red})$ are isomorphic as Poisson algebras.

This proves that M_{red} is a Poisson manifold. Finally, note that the coisotropic reduction admits as a special case the reduction with respect a Hamiltonian group action. In this case, the coisotropic submanifold is given by the preimage of a regular value of a momentum map. More precisely, consider a canonical action $\Phi : G \times M \to M$ generated by an ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$. If $\mu \in \mathfrak{g}^*$ is a regular value of J and is ad^*-invariant, then

$$C_\mu = J^{-1}(\mu) \subseteq M \tag{2.3}$$

is either empty or a coisotropic submanifold. Then the leaf space $C_\mu/\!\sim$ coincides with the orbit space C_μ/G (see e.g. [5]), so we get the reduced space of Theorem 2.2.

3. Pre-Hamiltonian actions

In this section we introduce a generalization of Hamiltonian actions in the setting of Poisson Lie groups acting on Poisson manifolds. For this reason we first recall some basic notions. A **Poisson Lie group** is a pair (G, π_G), where G is a Lie group and π_G is a multiplicative Poisson structure. The corresponding infinitesimal object is given by a **Lie bialgebra**, i.e. the Lie algebra $\mathfrak{g}$ corresponding to the Lie group G, equipped with the 1-cocycle,

$$\delta = \mathrm{d}_e\, \pi_G : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}. \tag{3.1}$$

If G is connected and simply connected there is a one-to-one correspondence between Poisson Lie groups and Lie bialgebras (known as Drinfeld's principle [12]). When $(\mathfrak{g}, \delta)$ is a Lie bialgebra, the 1-cocycle δ gives a Lie algebra structure on $\mathfrak{g}^*$, while the Lie bracket of $\mathfrak{g}$ gives a 1-cocycle δ^*on $\mathfrak{g}^*$, so that $(\mathfrak{g}^*, \delta^*)$ is also a Lie bialgebra. Thus, we can define the **dual Poisson Lie group** (G^*, π_{G^*}) as the (connected and simply connected) Poisson Lie group associated to the Lie bialgebra $(\mathfrak{g}^*, \delta^*)$. From now on we assume G to be connected and simply connected in order to get the one-to-one correspondence stated above.

Definition 3.1. An action of (G, π_G) on (M, π) is said to be a **Poisson action** if the map $\Phi : G \times M \to M$ is Poisson, that is

$$\{f \circ \Phi, g \circ \Phi\}_{G\times M} = \{f, g\}_M \circ \Phi, \qquad \forall f, g \in \mathscr{C}^\infty(M) \tag{3.2}$$

where the Poisson structure on $G \times M$ is given by $\pi_G \oplus \pi$.

It is evident that the above definition generalizes the notion of canonical action.

Definition 3.2 ([23, 24])**.** A **momentum map** for the Poisson action $\Phi : G\times M \to M$ is a map $J : M \to G^*$ such that

$$\varphi(\xi) = \pi^\sharp(J^*(\theta_\xi)), \tag{3.3}$$

where $\varphi : \mathfrak{g} \to \Gamma(TM)$ denotes the infinitesimal generator of the action, θ_ξ is the left invariant 1-form on G^* defined by the element $\xi \in \mathfrak{g} = (T_eG^*)^*$ and J^* is the cotangent lift of J.

Definition 3.3. Let $J : M \to G^*$ be a momentum map of the action Φ. Then,

(i) J is said to be G**-equivariant** if it is a Poisson map, i.e.

$$J_*\pi = \pi_{G^*}, \tag{3.4}$$

(ii) Φ is said to be a **Poisson Hamiltonian action** if it is a Poisson action induced by a G-equivariant momentum map.

This definition generalizes Hamiltonian actions in the canonical setting. Indeed, we notice that, if the Poisson structure on G is trivial, the dual G^* corresponds to the dual of the Lie algebra $\mathfrak{g}^*$ and the 1-form $J^*(\theta_\xi)$ is then exact. Thus, it recovers the usual definition of momentum map $J : M \to \mathfrak{g}^*$ for Hamiltonian actions in the canonical setting since $\varphi(\xi)$ is a Hamiltonian vector field. As pointed out by Ginzburg in [19], in many cases it is enough to consider the infinitesimal version of the G-equivariant momentum map, which is a map from the Lie bialgebra $\mathfrak{g}$ to the space of 1-forms on M. Recall that a Poisson structure π on a manifold M defines a Lie bracket $[\cdot, \cdot]_\pi$ on the space of 1-forms.

Definition 3.4. Let (M, π) be a Poisson manifold endowed with an action of a Poisson Lie group (G, π_G).

(1) A **PG-map** is a linear map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ such that
 (i) $\tilde{\varphi}_{[\xi,\eta]} = [\tilde{\varphi}_\xi, \tilde{\varphi}_\eta]_\pi$
 (ii) $\mathrm{d}\,\tilde{\varphi}_\xi = \tilde{\varphi} \wedge \tilde{\varphi} \circ \delta(\xi)$.

(2) Moreover, if $\tilde{\varphi}$ generates the action, that is

$$\varphi(\xi) = \pi^\sharp(\tilde{\varphi}_\xi), \tag{3.5}$$

we say that it is an **infinitesimal momentum map**.

The existence and uniqueness of the infinitesimal momentum map were studied in [19]. In particular, it was shown that an action of a compact group with $H^2(\mathfrak{g}) = 0$ admits an infinitesimal momentum map.

We are interested to study reduction for actions that admit an infinitesimal momentum map or just a PG-map.

Definition 3.5. An action of a Poisson Lie group on a Poisson manifold is said to be **pre-Hamiltonian** if it is generated by an infinitesimal momentum map.

It is important to remark that this notion is weaker than that of Poisson Hamiltonian action, as it does not reduce to the canonical one when the Poisson structure on G is trivial. In fact, in this case we only have that $\tilde{\varphi}_\xi$ is a closed form, but in general this form is not exact. However, as we will show, it turns out that every pre-Hamiltonian action is a Poisson action. Concrete examples of pre-Hamiltonian actions which are not Poisson Hamiltonian were provided in [19]. The study of the conditions in which the infinitesimal momentum $\tilde{\varphi}$ map determines the momentum map J can be found in [14].

Remark 3.6. Recall that a **Gerstenhaber algebra** (see [22]) is a triple $(A = \oplus_{i\in\mathbb{Z}} A^i, \wedge, [\ ,\ ])$ such that $(A, \wedge)$ is a graded commutative associative algebra, $(A = \oplus_{i\in\mathbb{Z}} A^{(i)}, [\ ,\ ])$, with $A^{(i)} = A^{i+1}$, is a graded Lie algebra, and for each $a \in A^{(i)}$ one has that $[a,\ ]$ is a derivation of degree i with respect to $\wedge$. A **differential Gerstenhaber algebra** $(A = \oplus_{i\in\mathbb{Z}} A^i, \mathrm{d}, \wedge, [\ ,\ ])$ is a Gerstenhaber algebra equipped with a differential d, that is a derivation of degree 1 and square zero of the associative product $\wedge$. One speaks of a **strong differential Gerstenhaber algebra** if, moreover, d is a derivation of the graded Lie bracket $[\ ,\ ]$. A **morphism of differential Gerstenhaber algebras** is a cochain map that respects the wedge product and the graded Lie bracket. It was proven in [22] that there is a one to one correspondence between Lie bialgebroids and strong differential Gerstenhaber algebras. Let (M, π) be a Poisson manifold and (G, π_G) a Poisson Lie group. Then by [22] one has that $(\wedge^\bullet\mathfrak{g}, \delta, \wedge, [\ ,\ ])$ and $(\Omega^\bullet(M), \mathrm{d}_{DR}, \wedge, [\ ,\ ]_\pi)$ are strong differential Gerstenhaber algebras. It is easy to check that the notion of infinitesimal momentum map can be rephrased as a morphism of differential Gerstenhaber algebras

$$\tilde{\varphi} : (\wedge^\bullet\mathfrak{g}, \delta, \wedge, [\ ,\ ]) \longrightarrow (\Omega^\bullet(M), \mathrm{d}_{DR}, \wedge, [\ ,\ ]_\pi). \tag{3.6}$$

However, notice that in general, despite being a morphism of differential Gerstenhaber algebras, an infinitesimal momentum map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ does not always correspond to a morphism of vector bundles $\mathfrak{g} \to T^*M$ and hence it does not necessarily induce a morphism of Lie algebroids from $\mathfrak{g}$ to T^*M.

3.1. **Properties of PG-maps.** The notion of a PG-map is crucial in order to prove a reduction theorem in this context. For this reason in this section we study some of its properties. In particular, we can prove that any PG-map (and hence any infinitesimal momentum map) defines a Lie bialgebroid morphism. Let us first recall the definitions related with Lie algebroids that we use in this paper.

Definition 3.7. Let $E \to M$ and $F \to N$ be two Lie algebroids. A **Lie algebroid morphism** is a bundle map $\Phi : E \to F$ such that

$$\Phi^* : (\Gamma(\wedge^\bullet F^*), \mathrm{d}^F) \to (\Gamma(\wedge^\bullet E^*), \mathrm{d}^E)$$

is a cochain map.

Definition 3.8. Assume that $E \to M$ is a Lie algebroid and that its dual vector bundle $E^* \to M$ also carries a structure of Lie algebroid. The pair (E, E^*) of Lie algebroids is a **Lie bialgebroid** if these differentials are derivations of the corresponding Schouten brackets, i.e. for any $X, Y \in \Gamma(E)$

$$\mathrm{d}_*[X, Y] = [\mathrm{d}_* X, Y] + [X, \mathrm{d}_* Y]. \tag{3.7}$$

It is important to mention that given a Lie bialgebroid (E, E^*), the Lie algebroid structure on E always induces a linear Poisson structure on E^* and viceversa. The most canonical example is given by the Lie bialgebroid (TM, T^*M) associated to a Poisson manifold M. In particular, given a Poisson manifold M, its tangent bundle carries a linear Poisson structure as shown in the following lemma.

Lemma 3.9 ([25, Prop. 10.3.12])**.** *Let (M, π_M) be a Poisson manifold. Then its tangent bundle TM has a linear Poisson structure π_{TM} defined by*

$$\pi^\sharp_{TM} \circ \alpha_M = k_M \circ T\pi^\sharp_M, \tag{3.8}$$

where $k_M : TTM \to TTM$ is the canonical involution of the double tangent bundle and $\alpha_M : TT^*M \to T^*TM$ is the Tulczyjew isomorphism [34, 35]*.*

Note that the linear Poisson structure π_{TM} on TM coincides with the standard complete lift of π_M.

We can now give the needed definition of a morphism of Lie bialgebroids.

Definition 3.10. A **Lie bialgebroid morphism** is a Lie algebroid morphism which is a Poisson map.

In order to prove that any PG-map $\tilde{\varphi}$ (see Def. 3.4), corresponds to a Lie bialgebroid morphism, we need to introduce a dual notion to that of PG-map.

Definition 3.11. Given a map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$, we can associate the map $c : TM \to \mathfrak{g}^*$ defined by

$$\langle c(X_m), \xi \rangle = \langle X_m, \tilde{\varphi}_\xi(m) \rangle, \tag{3.9}$$

for any $X_m \in T_mM$. If $\tilde{\varphi}$ is an infinitesimal momentum map we call c a **comomentum map**.

We are now ready to prove the announced result.

Proposition 3.12. *Let $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ be a PG-map. The associated map $c : TM \to \mathfrak{g}^*$ is a Lie bialgebroid morphism.*

Proof. From the definition it follows immediately that c is a morphism of vector bundles. Indeed, being a vector bundle over a point, $\mathfrak{g}^*$ has just one fiber, hence $\tilde{\varphi}$ sends fibers into fibers. Moreover, c is fiberwise linear, due to the linearity of $\tilde{\varphi}$. Finally, the pull-back $c^* : \Gamma(\wedge^\bullet \mathfrak{g}) \to \Gamma(\wedge^\bullet T^*M)$ is given by the the natural extension of the map $\tilde{\varphi}$ and hence it is a cochain map. Thus, c is a morphism of Lie algebroids. It remains to prove that the map $c^* : \mathscr{C}^\infty(\mathfrak{g}^*) \to \mathscr{C}^\infty(TM)$ is a Poisson map, i.e. $\{f, g\}_{\mathfrak{g}^*} \circ c = \{f \circ c, g \circ c\}_{TM}$. First, we consider f and g to be linear maps from $\mathfrak{g}^*$ to $\mathbb{R}$, so can denote them as

$$f = l_\xi, \quad g = l_\eta$$

for $\xi, \eta \in \mathfrak{g}$. For any $\xi \in \mathfrak{g}$, we now define

$$l_{\xi^\dagger} := l_\xi \circ c.$$

So, we aim to prove that

$$\{l_{\xi^\dagger}, l_{\eta^\dagger}\} = l_{[\xi,\eta]^\dagger}.$$

Using the definition of c it is evident that

$$l_{\xi^\dagger}(v_m) = \tilde{\varphi}_\xi(v_m),$$

for any $v_m \in T_mM$. Thus

$$l_{\xi^\dagger} = \tilde{\varphi}_\xi.$$

Hence,

$$\{l_{\xi^\dagger}, l_{\eta^\dagger}\} = \{\tilde{\varphi}_\xi, \tilde{\varphi}_\eta\} = \tilde{\varphi}_{[\xi,\eta]} = l_{[\xi,\eta]^\dagger}.$$

The extension to smooth functions can be easily done. In facts, we can immediately extend the result to polynomials and it is known that the space of polynomials is dense in the space of smooth functions. □

Given an action of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π), its infinitesimal generator is a map $\mathfrak{g} \to \Gamma(TM) : \xi \mapsto \varphi(\xi)$. Dualizing this map one gets a map $j : T^*M \to \mathfrak{g}^*$. In [37, Prop. 6.1], the author proves that the action is Poisson if and only if j is a Lie bialgebroid morphism. As a consequence we obtain the following result.

Proposition 3.13. *Let $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ be an infinitesimal momentum map. Then the action induced by $\tilde{\varphi}$ is a Poisson action.*

Proof. By dualizing (3.5) we obtain that in our case the dual of the infinitesimal action is

$$j = c \circ \pi^\sharp,$$

where c is the associated comomentum map defined by (3.9). From Prop. 3.12 we have that c is a Lie bialgebroid morphism. Moreover, it is well-known that $\pi^\sharp$ is a Lie bialgebroid morphism, so the composition j is a Lie bialgebroid morphism as well. The claim then follows by [37, Prop. 6.1]. □

It is important to remark that the above proposition implies that a pre-Hamiltonian action is always a Poisson action.

4. Reduction of the tangent bundle

In this section, using the techniques of coisotropic reduction recalled in Sec.2 and the properties of PG-maps, we prove a reduction theorem for the tangent bundle of a Poisson manifold (M, π) under the action of a Poisson Lie group. It is known that the tangent bundle of a Poisson manifold inherits a linear Poisson structure. We will show that a PG-map automatically produces a coisotropic submanifold of the tangent bundle. Thus, we obtain a reduced Poisson manifold. Furthermore, in the special case in which there is a pre-Hamiltonian action of a Poisson Lie group (G, π_G) on (M, π) we study the properties of the tangent lift of the action and this allows us to prove that the Poisson reduced space coincides with the G-orbit space as in the canonical setting. Finally, in the classic case of a Hamiltonian action on a Poisson manifold, we analyze the relation of the reduced tangent bundle $(TM)_{red}$ and the reduced manifold M_{red} produced by Theorem 2.2.

4.1. **Coisotropic and pre-Hamiltonian reduction.** Consider a Lie bialgebra $(\mathfrak{g}, \delta)$, a Poisson manifold (M, π) and let $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ be a PG-map. These ingredients are sufficient to obtain a coisotropic reduction. In Sec. 3.1, to a PG-map $\tilde{\varphi}$ we associated a dual map $c : TM \to \mathfrak{g}^*$ by (3.9) and we proved that it is a Poisson map.

The results on coisotropic reduction recalled in Sec. 2 can be immediately applied to this case. More explicitly, we can prove the following result.

Theorem 4.1. *Let (M, π) be a Poisson manifold endowed with a PG-map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$. Then $C := c^{-1}(0) \subseteq TM$ is a coisotropic submanifold, where $0 \in \mathfrak{g}^*$ is a regular value of c. Moreover, if C defines a simple foliation on TM, then the reduced manifold $(TM)_{red} = C/\sim$ has a Poisson structure.*

Proof. The fact that C is a coisotropic submanifold follows immediately by the fact that $\{0\}$ is a symplectic leaf in $\mathfrak{g}^*$ and from the fact that c is a Poisson map, as proved in Proposition 3.12.

To complete the proof it is enough to apply the coisotropic reduction Theorem 2.4 to our C. □

Now, we want to prove that the reduction in the case of a Pre-Hamiltonian action of a Poisson-Lie group gives rise to a special case of the coisotropic reduction obtained above. In the following we always assume the action to be free and proper.

Assume that we have an infinitesimal momentum map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$. The associated action is in general not Hamiltonian unless $\tilde{\varphi}_\xi$ is exact. However, we will see that if $\tilde{\varphi}_\xi$ is closed the lifted action on the tangent bundle is always Hamiltonian. In order to prove these results, we need some tools from the theory of derivations along maps [30].

A **tangent derivation** (that is, a derivation along τ_M^*, see [30]) of degree p is a linear operator $\mathrm{D} : \Omega^k(M) \to \Omega^{k+p}(TM)$ such that

$$\mathrm{D}(\omega_1 \wedge \omega_2) = \mathrm{D}\,\omega_1 \wedge \tau_M^*\omega_2 + (-1)^{kp}\tau_M^*\omega_1 \wedge \mathrm{D}\,\omega_2. \tag{4.1}$$

We define $i_T : \Omega^k(M) \to \Omega^{k-1}(TM)$ as the tangent derivation of degree -1 such that it is zero on functions and acts on any 1-form $\theta : M \to T^*M$ by

$$i_T\theta(v) = \langle \theta(\tau_M(v)), v \rangle, \tag{4.2}$$

for any $v \in TM$.

Remark 4.2. Notice that given $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ and $c : TM \to \mathfrak{g}^*$ we can express the map

$$\begin{aligned} \mathfrak{g} &\to \mathscr{C}^\infty(TM) \\ \xi &\mapsto c_\xi := c \circ \xi \end{aligned}$$

in terms of the tangent derivation i_T defined above. We get

$$c_\xi = i_T \tilde{\varphi}_\xi \tag{4.3}$$

Then, one easily obtains that on any k-form ω on M,

$$i_T\omega(w_1, \ldots, w_{k-1}) = \langle \omega(\tau_{TM}(w)), T\tau_M(w_1), \ldots, T\tau_M(w_{k-1})\rangle$$

for any $w_1, \ldots, w_{k-1} \in TTM$ such that $\tau_{TM}(w_1) = \ldots = \tau_{TM}(w_{k-1})$. If ω_1 is a k-form and ω_2 is an l-form, from (4.1) one has

$$i_T(\omega_1 \wedge \omega_2) = i_T\omega_1 \wedge \tau_M^*\omega_2 + (-1)^k \tau_M^*\omega_1 \wedge i_T\omega_2 \tag{4.4}$$

One can also define

$$\mathrm{d}_T \,\omega = i_T \,\mathrm{d}\,\omega + \mathrm{d}\, i_T\omega \tag{4.5}$$

It is easy to check that $\mathrm{d}_T : \Omega^k(M) \to \Omega^k(TM)$ is a tangent derivation of degree 0 and

$$\mathrm{d}_T(\omega_1 \wedge \omega_2) = \mathrm{d}_T\, \omega_1 \wedge \tau_M^*\omega_2 + (-1)^k \tau_M^*\omega_1 \wedge \mathrm{d}_T\, \omega_2. \tag{4.6}$$

Note that if ω is a k-form on a manifold M, then the k-form $\mathrm{d}_T\,\omega$ on TM is just the standard complete lift of ω. The theory of complete lifts of tensor field to the tangent bundle was developed in [38].

The following result is known but a proof is not easily available, so we provide one below.

Lemma 4.3. *For any 1-form $\theta : M \to T^*M$ on a manifold M, one has $T\theta : TM \to TT^*M$ and*

$$\alpha_M \circ T\theta = \mathrm{d}_T\, \theta. \tag{4.7}$$

Proof. In this proof we will make use of the Einstein summation convention. Let us take suitable local coordinate charts (q^i) in M and (q^i, v^j) in TM (with $i, j = 1, \ldots, n$). Then, the 1-form θ, seen as a map $\theta : M \to TM$ has the following coordinate expression

$$\theta(q) = (q^i, \theta_j(q)). \tag{4.8}$$

Hence for any $v \in TM$ of coordinates (q^i, v^j) one has

$$i_T\theta(v) = \theta_i(q)v^i.$$

Thus

$$\mathrm{d}\, i_T\theta(v) = (q^i, v^j, \partial_{q_i}\theta_j(q)v^j, \theta_j(q)). \tag{4.9}$$

On the other hand

$$\mathrm{d}\,\theta = \frac{1}{2}(\partial_{q_i}\theta_j - \partial_{q_j}\theta_i)dq^i \wedge dq^j.$$

Hence

$$i_T\,\mathrm{d}\,\theta(v) = (\partial_{q_j}\theta_i - \partial_{q_i}\theta_j)v^j dq^i. \tag{4.10}$$

Thus $\mathrm{d}_T\,\theta = i_T\,\mathrm{d}\,\theta + \mathrm{d}\,i_T\theta$ has the following local coordinate expression

$$\mathrm{d}_T\,\theta(v) = (q^i, v^j, \partial_{q_j}\theta_i(q)v^j, \theta_j(q)). \tag{4.11}$$

On the other hand, from (4.8) one has

$$T_q\theta(v) = v^i\,\mathrm{d}\,q^i + \partial_{q_i}\theta_j v^i\,\mathrm{d}\,v^j.$$

Hence as a map

$$T\theta(v) = (q^i, \theta_j(q), v^i, \partial_{q_i}\theta_j(q)v^i).$$

Now recall that (see e.g. [35])

$$\alpha_M(q^i, p_j, \dot{q}^h, \dot{p_k}) = (q^i, \dot{q}^h, \dot{p_k}, p_j).$$

Hence

$$\alpha_M \circ T\theta(v) = (q^i, v^j, \partial_{q_j}\theta_i(q)v^j, \theta_j(q)). \tag{4.12}$$

By comparing (4.11) and (4.12), the claim follows. □

Remark 4.4. Note that a similar result to Lemma 4.3 is [20, Theorem 2.1] that relates $i_T(T\theta)$ and $i_T(\mathrm{d}_T\,\theta)$ via k_M. Now, k_M is in a sense dual to α_M. So one could try to derive the lemma from the result in [20]. However, the duality between k_M and α_M is highly nontrivial, as it involves two different pairings (see e.g. [35]). Hence we think that it is easier to prove our claim directly, as we did above.

Given a pre-Hamiltonian action, the above results allow us to compute explicitly the infinitesimal generator of the tangent lift of the action, as can be seen in the following

Theorem 4.5. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson Lie group with infinitesimal momentum map $\tilde{\varphi}$.*

(i) *The infinitesimal generator φ^T of the tangent lift of Φ is given by*

$$\varphi^T(\xi) = X_{i_T\tilde{\varphi}_\xi} + \pi^\sharp_{TM} \circ i_T\mathrm{d}\,\tilde{\varphi}_\xi.$$

(ii) *If for each $\xi \in \mathfrak{g}$, one has $\mathrm{d}\,\tilde{\varphi}_\xi = 0$, then the lifted (infinitesimal) action on (TM, π_{TM}) is Hamiltonian, with fiberwise-linear momentum map defined by $c_\xi = i_T\tilde{\varphi}_\xi$.*

Proof. (i) Since $\tilde{\varphi}$ generates the action, we have the relation

$$\varphi(\xi) = \pi^\sharp_M \circ \tilde{\varphi}_\xi.$$

Moreover (see [20] or [25, p.365]),

$$\varphi^T(\xi) = k_M \circ T(\varphi(\xi)).$$

Now, by substituting the first relation in the second one, we obtain

$$\varphi^T(\xi) = k_M \circ T\pi^\sharp_M \circ T\tilde{\varphi}_\xi.$$

By using (3.8) and (4.7), we get

$$\begin{aligned}
\varphi^T(\xi) &= \pi^\sharp_{TM} \circ \alpha_M \circ T\tilde{\varphi}_\xi \\
&= \pi^\sharp_{TM} \circ \mathrm{d}_T\,\tilde{\varphi}_\xi \\
&= \pi^\sharp_{TM} \circ (\mathrm{d}\,i_T\tilde{\varphi}_\xi + i_T\,\mathrm{d}\,\tilde{\varphi}_\xi) \\
&= X_{i_T\tilde{\varphi}_\xi} + \pi^\sharp_{TM} \circ i_T\,\mathrm{d}\,\tilde{\varphi}_\xi.
\end{aligned}$$

(ii) Clearly, if $\mathrm{d}\,\tilde{\varphi}_\xi = 0$, we get $\varphi^T(\xi) = X_{i_T\tilde{\varphi}_\xi}$.

□

It is clear that if $\mathrm{d}\,\tilde{\varphi}_\xi = 0$ for any $\xi \in \mathfrak{g}$, then we can reduce the tangent bundle by using the well-known Theorem 2.2 of reduction of Poisson manifolds, since in Theorem 4.5 we proved that in this case the tangent lift of the action is Hamiltonian. In other words, the reduction procedure obtained above recovers the reduction of Poisson manifolds in the specific case in which the infinitesimal momentum map associates a closed form to any element of the Lie bialgebra. In particular, this happens in the case of a symplectic action on a symplectic manifold (M, ω_M). Then, the tangent bundle is also symplectic, with the symplectic form given by $d_T\omega_M$.

Corollary 4.6. *Let $\mathfrak{g} \to \mathfrak{X}(M)$ be a symplectic action of a Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω_M). Then, the lifted action on $(TM, d_T\omega_M)$ is Hamiltonian, with fiberwise-linear momentum map $c : TM \to \mathfrak{g}^*$ defined by*

$$c_\xi = i_T(i_{\varphi(\xi)}\omega_M).$$

Proof. Under the above assumptions we have that the action is clearly pre-Hamiltonian with infinitesimal momentum map $\tilde{\varphi} : \mathfrak{g} \to \Omega^1(M)$ given by

$$\tilde{\varphi}_\xi = i_{\varphi(\xi)}\omega_M.$$

Moreover, $\mathscr{L}_{\varphi(\xi)}\omega_M = 0$ implies $\mathrm{d}\, i_{\varphi(\xi)}\omega_M = 0$ since ω_M is closed. □

Theorem 4.5 allows us to show that, in the case of a (free and proper) pre-Hamiltonian $G-$action, the reduced Poisson manifold $(TM)_{red} = C/\!\sim$ of Theorem 4.1 is the orbit space of the lifted action of G on $C \subseteq TM$.

Theorem 4.7. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson Lie group with infinitesimal momentum map $\tilde{\varphi}$ and comomentum map c. Then the reduced tangent bundle is given by $(TM)_{red} = C/G$. Moreover $(TM)_{red} \to M/G$ is a Lie algebroid.*

Proof. Let $\{e^i\}_{i=1,\ldots,n}$ be a basis of $\mathfrak{g}^*$ and $c_i : TM \to \mathbb{R}$ the components of c. Thus,

$$c = \sum_i c_i e^i.$$

Since $C = c^{-1}(0)$ and $\mathscr{I}_C$ is the set of functions vanishing on C, then by [5, Lemma 5] any $f \in \mathscr{I}_C$ can be written as

$$f = \sum_i f^i c_i, \tag{4.13}$$

where

$$f^i : TM \to \mathbb{R}.$$

Consider the inclusion $i : C \to TM$ and a Hamiltonian vector field X_f on TM (recall that they span the characteristic distribution on C). From (4.13) we have

$$i^*X_f = \sum_i i^*(f^i X_{c_i} + c_i X_{f^i}),$$

by the Leibniz rule. The term $i^*(c_i X_{f^i})$ is zero because c_i vanishes on C. So we get

$$i^* X_f = \sum_i i^*(f^i X_{c_i}). \tag{4.14}$$

From Theorem 4.5 and (4.3), we have

$$\varphi^T(e_i) = X_{i_T \tilde{\varphi}_i} + \pi^\sharp_{TM} \circ i_T \mathrm{d}\, \tilde{\varphi}_i = X_{c_i} + \pi^\sharp_{TM} \circ i_T \mathrm{d}\, \tilde{\varphi}_i, \tag{4.15}$$

where $\tilde{\varphi}_i := \tilde{\varphi}(e_i)$. We have

$$\delta(e_i) = \sum_{j<k} \gamma_i^{jk} e_j \wedge e_k,$$

where γ_i^{jk} are some real constants. Now, using the property $\mathrm{d}\, \tilde{\varphi}_\xi = \tilde{\varphi} \wedge \tilde{\varphi} \circ \delta(\xi)$ we can write

$$i_T \mathrm{d}\, \tilde{\varphi}_i = \sum_{j<k} \gamma_i^{jk} i_T(\tilde{\varphi}_j \wedge \tilde{\varphi}_k).$$

Hence, from (4.3) and (4.4) we get

$$i_T \mathrm{d}\, \tilde{\varphi}_i = \sum_{j<k} \gamma_i^{jk} (c_j \wedge \tau_M^* \tilde{\varphi}_k - \tau_M^* \tilde{\varphi}_j \wedge c_k).$$

Thus, since the c_i s' are functions, we have

$$\pi^\sharp_{TM}(i_T \mathrm{d}\, \tilde{\varphi}_i) = \sum_{j<k} \gamma_i^{jk} (c_j \pi^\sharp_{TM}(\tau_M^* \tilde{\varphi}_k) - c_k \pi^\sharp_{TM}(\tau_M^* \tilde{\varphi}_j)).$$

From (4.15), by using this relation we get

$$i^* \varphi^T(e_i) = i^*(X_{c_i} + \pi^\sharp_{TM}(i_T \mathrm{d}\, \tilde{\varphi}_i)) = i^* X_{c_i} \tag{4.16}$$

because the functions c_i s' vanish on C. Substituting (4.16) in (4.14) we have

$$i^* X_f = \sum_i i^*(f^i) \cdot i^* \varphi^T(e_i).$$

We have proved that the leaves of the characteristic distribution are the G-orbits.

It remains to prove that $(TM)_{red} \to M/G$ is a Lie algebroid. over M/G First note that $C = c^{-1}(0) \to M$ is a Lie subalgebroid of the tangent bundle, as c is a morphism of Lie (bi)algebroids. Moreover, the free and proper action of G on M lifts to free and proper action on TM by Lie algebroid automorphisms. Thus, since the lifted action restricts to an action by Lie algebroid automorphisms on C, one gets a quotient Lie algebroid $C/G \to M/G$ (see e.g. [25, 26]). □

Remark 4.8. As an example, let us consider the dressing action $G \times G^* \to G^*$, which is Poisson Hamiltonian with momentum map $J = \mathrm{id}$. Thus, using the linearization $TG^* \cong G^* \times \mathfrak{g}^*$ and the definition of the comomentum map (3.9), we get

$$c = \mathrm{pr}_{\mathfrak{g}^*} : G^* \times \mathfrak{g}^* \to \mathfrak{g}^*.$$

Hence, in this case the reduction of the tangent bundle gives as a result the space of orbits of the dressing action:

$$(TG^*)_{red} = G^*/G.$$

4.2. **Relation with the Hamiltonian case.** Let us consider the particular case in which the pre-Hamiltonian action is Hamiltonian, so we have a momentum map $J : M \to \mathfrak{g}^*$ and $\tilde{\varphi}_\xi = \mathrm{d}\, J_\xi$ (for instance, this occurs if $\tilde{\varphi}_\xi = J^*(\theta_\xi)$ and $\pi_G = 0$). As recalled in Section 2, in this case we have the well-known reduction Theorem 2.2 which gives a reduced Poisson manifold $M_{red} = J^{-1}(0)/G$. The following theorem gives the relation between M_{red} and $(TM)_{red}$.

Theorem 4.9. *Let (M, π) be a Poisson manifold endowed with a Hamiltonian action of a Lie group G and assume that $0 \in \mathfrak{g}^*$ is a regular value for the momentum map $J : M \to \mathfrak{g}^*$. Then, we have $(TM)_{red} = (\mathrm{d}_T\, J)^{-1}(0)/G$. Moreover, there is a connection dependent isomorphism of vector bundles*

$$(TM)_{red}|_{M_{red}} \cong T(M_{red}) + \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the associated bundle to the principal bundle $J^{-1}(0) \to J^{-1}(0)/G$ by the adjoint action of G on $\mathfrak{g}$.

Proof. Recall that

$$M_{red} = J^{-1}(0)/G,$$

where $J : M \to \mathfrak{g}^*$ and

$$(TM)_{red} = c^{-1}(0)/G,$$

where $c : TM \to \mathfrak{g}^*$ is the comomentum map associated to $\tilde{\varphi}$. Moreover, we have

$$\tilde{\varphi}_\xi = \mathrm{d}\, J_\xi,$$

for each $\xi \in \mathfrak{g}$. Hence $c_\xi : TM \to \mathbb{R}$ is given by

$$c_\xi = i_T \tilde{\varphi}_\xi = i_T\, \mathrm{d}\, J_\xi = \mathrm{d}_T\, J_\xi. \tag{4.17}$$

We can extend the action of d_T to act on $\mathfrak{g}^*$-valued functions such as J in an obvious way, giving as a result $\mathrm{d}_T\, J : TM \to \mathfrak{g}^*$. Then, from (4.17) we obtain $c = \mathrm{d}_T\, J$ and hence

$$(TM)_{red} = (\mathrm{d}_T\, J)^{-1}(0)/G. \tag{4.18}$$

Now, we have

$$(\mathrm{d}_T\, J)^{-1}(0)|_{J^{-1}(0)} = T(J^{-1}(0)). \tag{4.19}$$

Indeed, if $v \in T(J^{-1}(0)) \subset TM$ then $J(\tau_M(v)) = 0$ and $\langle \mathrm{d}\, J(\tau_M(v)), v\rangle = 0$. Hence

$$(\mathrm{d}_T\, J)(v) = i_T\, \mathrm{d}\, J(v) = 0.$$

Conversely, if $v \in (\mathrm{d}_T\, J)^{-1}(0)$ and $J(\tau_M(v)) = 0$ then $\tau_M(v) \in J^{-1}(0)$ and

$$0 = (\mathrm{d}_T\, J)(v) = i_T\, \mathrm{d}\, J(v) = \langle \mathrm{d}\, J(\tau_M(v)), v\rangle.$$

Thus, we get $v \in T(J^{-1}(0))$.

We use now the following fact. Since the lifted action of G on TM is free and proper, by [11, Lemma 2.4.2] one has a connection dependent isomorphism

$$(TM)/G \cong T(M/G) + \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the associated bundle to the principal bundle $M \to M/G$ by the adjoint action of G on $\mathfrak{g}$. If we apply the above result to $J^{-1}(0)$, we get

$$(TJ^{-1}(0))/G \cong T(J^{-1}(0)/G) + \tilde{\mathfrak{g}} = T(M_{red}) + \tilde{\mathfrak{g}}$$

and hence by (4.19) we obtain

$$(\mathrm{d}_T\, J)^{-1}(0)|_{J^{-1}(0)}/G \cong T(M_{red}) + \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the associated bundle to the principal bundle $J^{-1}(0) \to J^{-1}(0)/G$ by the adjoint action of G on $\mathfrak{g}$. On the other hand, from (4.18) we have

$$(TM)_{red}|_{M_{red}} = (\mathrm{d}_T\, J)^{-1}(0)/G|_{J^{-1}(0)/G} = (\mathrm{d}_T\, J)^{-1}(0)|_{J^{-1}(0)}/G,$$

since the tangent projection is equivariant with respect to the considered group actions. Hence we conclude that

$$(TM)_{red}|_{M_{red}} \cong T(M_{red}) + \tilde{\mathfrak{g}}.$$

□

This theorem shows that in case of a Hamiltonian action our reduced tangent bundle is closely related to the tangent bundle of the classical Marsden-Ratiu reduced manifold.

Note that a part of Theorem 4.9 in the special case of symplectic manifolds was already proved in [18, Theorem 4.1].

5. Integration of the reduced tangent bundle

In this section we give an interpretation of the reduced tangent bundle in terms of symplectic groupoids. In particular, we consider the case of a symplectic action of a Lie group G on a symplectic manifold M. On the one hand, we have shown that in this case the lifted action on the tangent bundle TM is Hamiltonian so that we obtain a reduced tangent bundle $(TM)_{red}$ which is a symplectic manifold. On the other hand, we can apply to our case the results of [29] and [17] that hold for a canonical action on an integrable Poisson manifold (M, π). Now, every symplectic manifold is integrable and the corresponding symplectic groupoid that can be identified with the fundamental groupoid $\Pi(M)$ of M. Hence, the action of G on M can be lifted to a Hamiltonian action on the fundamental groupoid $\Pi(M) \rightrightarrows M$. This implies that the symplectic groupoid $\Pi(M)$ can be reduced via Marsden-Weinstein procedure to a new symplectic groupoid $(\Pi(M))_{red} \rightrightarrows M/G$. We prove that our reduced tangent bundle $(TM)_{red}$ is the Lie algebroid corresponding to the reduced symplectic groupoid $(\Pi(M))_{red}$.

First, let us recall the needed results from [17].

Theorem 5.1 ([17])**.** *Let $G \times M \to M$ a free and proper canonical action on an integrable Poisson manifold (M, π). There exists a unique lifted action of G on $\Sigma(M) \rightrightarrows M$ by symplectic groupoid automorphisms. This lifted action is free and proper and Hamiltonian. Let $J : \Sigma(M) \to \mathfrak{g}^*$ denote its momentum map. Then, the reduced symplectic groupoid, given by*

$$(\Sigma(M))_{red} = J^{-1}(0)/G$$

is a symplectic groupoid integrating M/G.

Note that the symplectic form on $(\Sigma(M))_{red}$ allows us to identify the Lie algebroid $A((\Sigma(M))_{red})$ with the cotangent Lie algebroid $T^*(M/G)$.

We will also use the following well-known result on the cotangent bundle reduction.

Theorem 5.2 ([1])**.** *Given a free and proper action of a Lie group G on M, the cotangent lift of the action is Hamiltonian with momentum map given by*

$$\langle j(\alpha_m), \xi \rangle = \alpha_m(\varphi_\xi(m)),$$

for any $\alpha_m \in T^*_m M$*,* $\xi \in \mathfrak{g}$*. Moreover, we have*

$$(T^*M)_{red} \cong T^*(M/G).$$

The following lemma shows that in the case of a symplectic action on a symplectic manifold (M, ω), due to the isomorphism $\omega^\flat : TM \to T^*M$ induced by ω, the reduced tangent bundle produced by Corollary 4.6 is isomorphic to the classical reduced cotangent bundle.

Lemma 5.3. *Given a symplectic action* $G \times M \to M$ *of a Lie group* G *on a symplectic manifold* (M, ω) *we have*

$$(TM)_{red} \cong (T^*M)_{red}.$$

Proof. The momentum map $j : T^*M \to \mathfrak{g}^*$ of the cotangent lift of the action is characterized by

$$j_\xi(\alpha_m) = \alpha_m(\varphi_\xi(m)),$$

for any $m \in M$, $\alpha_m \in T^*_m M$ and $\xi \in \mathfrak{g}$. Hence, by writing $\alpha_m = \omega^\flat X_m$ with $X_m \in T_m M$ we get

$$j_\xi(\omega^\flat X_m) = (i_{X_m}\omega)_m(\varphi_\xi(m)) = \omega_m(X_m, \varphi_\xi(m)).$$

On the other hand, by Corollary 4.6 the momentum map $c : TM \to \mathfrak{g}^*$ of the tangent lift of the action is characterized by

$$c_\xi(X_m) = i_T(i_{\varphi(\xi)}\omega)(X_m) = (i_{\varphi(\xi)}\omega)_m(X_m) = -\omega_m(X_m, \varphi_\xi(m)).$$

Thus we conclude that

$$c = -j \circ \omega^\flat.$$

As a consequence,

$$j^{-1}(0) = -\omega^\flat(c^{-1}(0)).$$

Since the symplectic form is G-invariant, we conclude that

$$j^{-1}(0)/G \cong c^{-1}(0)/G.$$

□

Note that Lemma 5.3 in the particular case when M is a canonical tangent bundle and the G-action is a cotangent lift was already proved in [18, Lemma 4.2].

These results allow us to prove that in the symplectic action of a Lie group on a symplectic manifold the reduced tangent manifold $(TM)_{red}$ is the Lie algebroid of the reduced symplectic groupoid $(\Pi(M))_{red} \rightrightarrows M/G$.

Theorem 5.4. *Given a free and proper symplectic action of a Lie group* G *on a symplectic manifold* (M, ω)*, we have*

$$A((\Pi(M))_{red}) \cong (TM)_{red}.$$

Proof. By Corollary 4.6, we know that the lifted action to TM is Hamiltonian with momentum map c. Thus we can perform the reduction of TM obtaining a reduced tangent bundle

$$(TM)_{red} = c^{-1}(0)/G$$

endowed with a symplectic structure.

From Theorem 5.1, we know that the symplectic action can be lifted to a Hamiltonian action on $\Sigma(M)$ that in this case can be identified with $\Pi(M)$ and the reduced symplectic groupoid is

$$(\Pi(M))_{red} := J^{-1}(0)/G \rightrightarrows M/G,$$

where $J : \Pi(M) \to \mathfrak{g}^*$ is the momentum map of the lifted action. In this case the base manifold M/G is symplectic. Moreover, we have

$$A((\Pi(M))_{red}) = T^*(M/G). \tag{5.1}$$

Now, by Theorem 5.2, we have

$$A((\Pi(M))_{red}) \cong (T^*M)_{red}. \tag{5.2}$$

Hence, by using Lemma 5.3 we get

$$A((\Pi(M))_{red}) \cong (TM)_{red}.$$

□

Note that if (M, ω) is simply connected the corresponding symplectic groupoid is the pair groupoid:

$$M \times \bar{M} \rightrightarrows M,$$

with symplectic structure $\omega \oplus (-\omega)$. Thus, we obtain the following result.

Corollary 5.5. *Let M be symplectic and simply connected and let $G \times M \to M$ be a symplectic action. Then,*

$$A((M \times \bar{M})_{red}) \cong (TM)_{red}.$$

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